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THE SIZE OF THE END ZONE AND THE PROPAGATION VELOCITY OF A DISPLACEMENT JUMP†

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Asymptotic expressions are obtained for the end zone of a propagating displacement jump. A formula is presented for the size of the end zone when there is linear weakening in it which, when used in Novozhilov's structural criterion, provides an answer to the question as to why the propagation velocity of normal or shear displacement jump, which is conventionally observed under mechanical loading in experiments and during earthquakes, is less than the Rayleigh wave velocity.

In the conclusion to the monograph [1], the "limitation of the crack velocities observed in experiments" was designated as being among the unsolved problems and, earlier ([1], p. 264), it was mentioned that "... in the case of a bounded critical stress intensity factor, the crack propagation velocity must tend to the Rayleigh wave velocity. Actually, as experiments show, under conditions of mechanical loading the limit velocity of a crack turns out to be much lower than the Rayleigh wave velocity". Recent seismograph records, which were directly on the rupture surface accompanying the Chi-Chi earthquake on Taiwan (20 September 1999) [2], confirm this fact.

The search for an answer to the question raised in [1] is still urgent since a transition through the Rayleigh wave is reproduced in the numerical simulation of rupture propagation (for example, see [3]). This transition was also obtained during a simulation of the Chi-Chi earthquake [4], which contradicts the above-mentioned data from direct observations.

All of this prompts one to return once more to the problem formulated in [1]. The results presented below were obtained from an analysis of the mechanism of the Chi-Chi earthquake. They serve as an explanation for the fact that rupture velocity observed accompanying mechanical loading is lower than the Rayleigh wave velocity. It is assumed here that the rupture velocity does not exceed the velocity of transverse waves.

Section 1 contains an extension of the asymptotic relations obtained previously for a static problem to a problem concerning a propagation displacement jump (a moving rupture). A comparison of the critical size of the weakening zone at the initial disruption with its size during the propagation of the rupture is presented in Section 2. It is established that the asymptotic formula is already applicable at distances comparable with the initial size of the zone of weakening. In Section 3, it is shown, on the basis of these conclusions, that it follows from the structure – time criterion [5–7] that the maximum rupture velocity is lower than the Rayleigh wave velocity. Branching, discontinuities, supersonic rupture propagation and other mechanisms for the absorption of excess energy, which may arise on account of the restricted size of the end zone, are not discussed here.

Treatment of the case when the rupture velocity, while being lower than the velocity of bulk waves, is found to be greater than the velocity of transverse waves, requires a more complex analysis. A brief remark about this case is made at the end of the paper.

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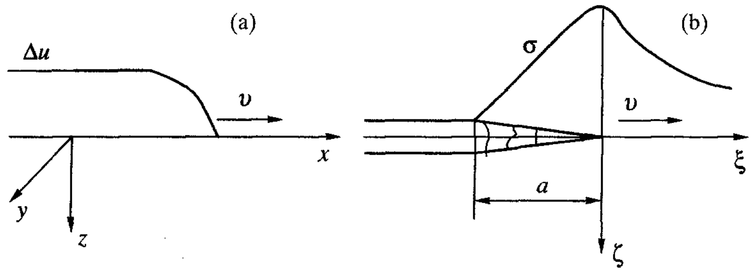

Fig. 1

1. ASYMPTOTIC RELATIONS FOR THE END ZONE OF A PROPAGATING DISPLACEMENT JUMP

Suppose the front of the displacement jump at the instant being considered has a velocity υ and has advanced by a sufficiently large distance l such that the field in the neighbourhood of the front can be considered as being a locally stationary field (in the next section, we shall estimate when this assumption is acceptable). We direct the x axis along the direction of motion of the rupture, the y axis along the front and the z axis perpendicular to the plane of the rupture so that the x, y and z axes form a right-handed system (Fig. 1a). For simplicity, we shall assume that the displacement vector behind the front has a discontinuity in only one of its components $\Delta u_x = u_x^+ - u_x^-$, $\Delta u_y = u_y^+ - u_y^-$ or $\Delta u_z = u_z^+ - u_z^-$. The stresses at the rupture and ahead of it are continuous: $\sigma_{zx}^+ = \sigma_{zx}^- = \sigma_{zx}$, $\sigma_{zy}^+ = \sigma_{zy}^- = \sigma_{zy}$, $\sigma_{zz}^+ = \sigma_{zz}^- = \sigma_{zz}$. We shall assume that they are equal to zero behind the front. The more general case of constant, non-zero stresses, as usual, is included as a superpositioning (see [8] for example).

In the neighbourhood of the tip of the rupture, the elastic field is steady in the system of coordinates $\xi = x - \upsilon t$, $\eta = y$, $\zeta = z$ which moves together with it (see Fig. 1b). Therefore, in order to obtain the asymptotic relations describing this field, we use Galin's steady-state solution [9] (see also [10, p. 120]) which we will represent in the form

$$-\frac{d\Delta u_i}{d\xi} = C_i \int_{-\infty}^{\infty} \frac{\sigma_{zi}(\zeta)d\zeta}{\zeta - \xi}, \quad i = x, z, y \tag{1.1}$$

where

$$C_i = \frac{1}{\pi\mu} f_i(p), \quad p = \frac{1}{\upsilon}$$

$$f_x(p) = -2p\beta\frac{\sqrt{p^2\beta^2 - 1}}{\Delta_R}, \quad f_z(p) = -2\frac{\beta}{\alpha}p\beta\frac{\sqrt{p^2\alpha^2 - 1}}{\Delta_R}, \quad f_y(p) = \frac{2p\beta}{\sqrt{p^2\beta^2 - 1}}$$

$$\Delta_R = (2p^2\beta^2 - 1)^2 - 4\frac{\beta}{\alpha}p^2\beta^2\sqrt{p^2\alpha^2 - 1}\sqrt{p^2\beta^2 - 1}$$

μ is the shear modulus, α and β are the velocities of the bulk and transverse waves and the equality $\Delta_R = 0$ defines the Rayleigh wave υ_R. In the case when Poisson's ration $\nu = 0.25$, we have $\alpha/\beta = \sqrt{3}$, $\upsilon_R = 0.9184\beta$.

There is no displacement jump ahead of the wave front. Subject to this condition, the key asymptotic formula for the end zone [11–13]

$$-\Delta u_i(\xi) = C_i \int_{-a}^{0} L(\xi, \zeta)\sigma_{zi}(\zeta)d\zeta - C_i k_i \sqrt{-2\pi\xi}; \quad L(\xi, \zeta) = \ln\left|\frac{\sqrt{-\xi} + \sqrt{-\zeta}}{\sqrt{-\xi} - \sqrt{-\zeta}}\right| \tag{1.2}$$

follows from relations (1.1), where k_i is the stress intensity factor (SIF): of the shear transverse ($i = x$), the normal ($i = z$) or the shear longitudinal ($i = y$) stresses. We stress that the SIF is due to the external loads rather than to the stresses in the end zone. In order that the stresses should be finite (and continuous) in the rupture front, the condition

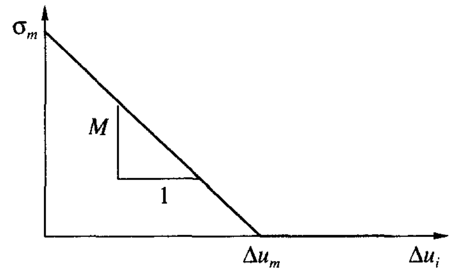

Fig. 2

$$\int_{-a}^{0} \frac{\sigma_{zi}(\zeta)d\zeta}{\sqrt{-\zeta}} = \sqrt{\frac{\pi}{2}}k_i \tag{1.3}$$

must be satisfied.

In the case of a specified law for the interaction between the sides of the rupture in the end zone, Eqs (1.2) and (1.3) determine the stress distribution and the displacement jump distribution in it and the size a of this zone. In the case of the generally used model of linear weakening see [3, 4, 12–17] for example), the relation between Δu_i and σ_{zi} has the form shown in Fig. 2. Its analytical expression contains the limit stress σ_m, which is withstood by the material and the weakening modulus M which has the dimensions [stress/length]:

$$\sigma_{zi} = \begin{cases} \sigma_m - M\Delta u_i, & \Delta u_i \le \Delta u_m \\ 0, & \Delta u_i \ge \Delta u_m \end{cases} \tag{1.4}$$

Substituting expression (1.4) into relations (1.2) and (1.3) normalizing

$$\sigma = \sigma_{zi}/\sigma_m, \quad \xi' = -\xi/a, \quad \zeta' = -\zeta/a$$

we obtain

$$\sigma(\xi') - \lambda\int_0^1 L'(\xi', \zeta')\sigma(\zeta')d\zeta' = 1 - \lambda\frac{k_i}{\sigma_m}\sqrt{\frac{2\pi\xi'}{a}} \tag{1.5}$$

$$\int_0^1 \frac{\sigma(\zeta')d\zeta'}{\sqrt{\zeta'}} = \sqrt{\frac{\pi}{2a}}\frac{k_i}{\sigma_m} \tag{1.6}$$

where

$$L'(\xi', \zeta') = L(-\xi', -\zeta'), \quad \lambda = C_i Ma \tag{1.7}$$

An analysis of Eqs (1.5) and (1.6) [12] shows that a solution having a physical meaning only exists for values of λ which do not exceed the critical value $\lambda_C = 0.4655$. When $\lambda = \lambda_C$, the stress vanishes when $\xi = -a$ and an increment in the free rupture surface corresponds to it. This means that progress of the rupture occurs when $\lambda = \lambda_C$. Taking into account the second equality of (1.7), we obtain that the condition

$$C_i Ma_C = \lambda_C \tag{1.8}$$

is satisfied during the propagation of the rupture, where a_C is the critical size of the zone of weakening in the case of the front propagation velocity υ.

The relation [12]

$$\frac{\pi}{2}C_i k_{ic}^2 \frac{M}{\sigma_m^2} = 1 \tag{1.9}$$

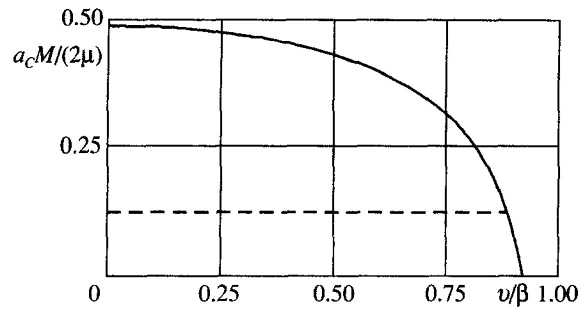

Fig. 3

corresponds to the condition (1.8) being satisfied and, in the dynamic problem being considered here, the quantity k_{ic} has the meaning of the critical value of the dynamic SIF. While this is not a constant of the material, it does depend on the rupture velocity. Only the critical value of the static SIF serves as a characteristic of the material.

Actually, in the static case, on taking the limit $\upsilon \to 0$ in the expressions for C_i, we have

$$C_{x0} = C_{z0} = 2(1-\nu)/(\pi\mu), \quad C_{y0} = 2/(\pi\mu)$$

where ν Poisson's ratio. We will now take account of the fact that, in the static case, the influx of energy per unit area $-dE/dS$ is equal to $(1-\nu)k_{x0}^2/(2\mu)$, $(1-\nu)k_{z0}^2/(2\mu)$ and $k_{y0}^2/(2\mu)$ for displacement jumps in the x-, z- and y-directions respectively (see [1, 10] for example). Condition (1.9), which strictly corresponds to the number of equation (1.5), takes the form of the Griffith–Orovan condition: $-dE/dS = 2\gamma$, where $2\gamma = \sigma_m^2/(2M)$. In Fig. 2, the area under the weakening line corresponds to 2γ. Hence, according to the Griffith–Orovan criterion, 2γ is the maximum energy absorbed per unit area during the rupture of the material. The energy 2γ and, consequently, the critical values of the static SIF, which are associated with it through the relations presented, serve as characteristics of the strength of the material.

We now turn to the dynamic condition in the form of (1.8). To be specific, we will consider it in the case of shear rupture: $\Delta u_x \neq 0$ when $\xi < 0$. We will write equality (1.8) in a form which determines the dimensionless length $a_C M/(2\mu)$ of the zone where there is weakening:

$$a_C\frac{M}{2\mu} = f_{xc}(p); \quad f_{xc}(p) = \frac{\pi}{2}\lambda_C\frac{\Delta_R}{-2p\beta\sqrt{p^2\beta^2-1}} \tag{1.10}$$

For a given Poisson's ratio ν, the function $f_{xc}(p)$ depends solely on the normalized rupture velocity υ/β.

A graph of the dimensionless length $a_C M/(2\mu)$ of the end zone where there is weakening, constructed for the usual value $\nu = 0.25$, is shown in Fig. 3. It can be seen that the size of the zone tends to zero in the case of a rupture velocity which tends to the Rayleigh wave velocity ($\upsilon_R/\beta = 0.9184$). The end zone disappears in the limit when $\upsilon = \upsilon_R$. Such a situation is unreal since, in order for the material to separate, it is necessary that a zone where this occurs should be preserved and, at least, a certain domain with a size of the order of the dimensions of the structural elements which area fractured. The corresponding constraint on the velocity of the rupture wave is discussed in Section 3.

2. THE DOMAIN OF APPLICABILITY OF THE ASYMPTOTIC FORMULA FOR THE END ZONE

We will now investigate how far a rupture must advance in order that the asymptotic formula for the size of the end zone can be confidently used. To do this, we use the initial rupture condition obtained in [18] from a consideration of a segment of length $2l_0$, on the whole surface of which the maximum stress σ_m is attained: the sides of this segment interact along its whole length in accordance with relation (1.4), which corresponds to linear weakening. Hence, in the case being considered, the "end" domains of weakening have a length $a_{C0} = l_0$ and they extend to the middle of the segment where they join. The critical value a_{C0}, obtained by a numerical solution of the problem, is determined by the condition

$$\frac{\chi+1}{2\mu}Ma_{C0} = 1.158$$

where χ is the Muskhelishvili parameter. In the case of plane deformation, $\chi = 3 - 4\nu$, and this relation can be written in the form, which is analogous to (1.10)

$$\frac{M}{2\mu}a_{C0} = \frac{1.158}{4(1-\nu)} \tag{2.1}$$

For $\nu = 0.25$, which has been used to construct the graph in Fig. 3, the right-hand side equality (2.1) is equal to 0.386. The analogous static value for the right-hand side of the asymptotic formula (1.10) is equal to $a_C(0)M/(2\mu) = 0.485$. It follows from this that the asymptotic formula (1.10) is already applicable when there is a moderate distance l between the end of the rupture and the middle of the rupture ($l \geq l_0$) and, even when $l = l_0 = a_{C0}$, the error is only 25.6%.

3. VELOCITY CONSTRAINT FOLLOWING FROM THE STRUCTURAL CRITERION

According to Novozhilov's structural criterion [5], a crack develops when the stress in an element, with a specified characteristic size, in the neighbourhood of the crack tip attains a limit value. The characteristic size is of the order of the size of the structural element which is being fractured. This size is clearly different for different structural levels. Nevertheless, it remains of the same order of magnitude in the case of observed macroscopic ruptures. This shows up particularly clearly in the case of a generalized structural criterion for dynamic problems in the form of a structural – time criterion [6, 7]. It has been shown that even the use of a constant value of the characteristic size, when appropriate account is taken of the duration of the loading pulse, enables one to describe crack propagation in a wide range of dynamic experiments. It is therefore justified to impose this physically obvious constraint on the size of the end zone. We note that this constraint was not imposed in the case considered in [3, 4], when the calculations gave a transition through the Rayleigh velocity.

In accordance with what has been said, we shall assume that, although the size of the end zone may be reduced in dynamic calculations, it cannot be reduced to zero: it remains of the order of the characteristic size of the structural elements which are being fractured. In the case of the normalized critical sizes shown in Fig. 3, the boundary corresponding to a quarter of the static size is shown by the dashed line. For this line, the maximum velocity is equal to $0.965\upsilon_R$.

It follows from the graph in Fig. 3 that a constraint on the size of the end zone leads to a constraint on the rupture velocity which prevents it from reaching the Rayleigh wave velocity. Consideration of the time a_C/υ for which the pulse acts, as in customary when the structure – time criterion [6, 7] is used, imposes an additional restriction on the velocity of the rupture wave. Hence, the maximum permissible size of the end zone controls the maximum possible rupture velocity. This situation, in conjunction with formula (1.10) and the graph in Fig. 3, constitutes the main result of this paper.

In conclusion, we recall that the results which have been presented were obtained on the assumption that the rupture velocity is lower than the velocity of transverse waves ($\upsilon < \beta$). It is impossible to reproduce the investigation of the opposite case ($\beta < \upsilon < \alpha$), carried out by the author, within the framework of this brief communication. We merely note that, in the case of a transverse displacement jump Δu_x, an initial relation of the type of (1.1) can also be used when $\beta < \upsilon < \alpha$. In this case, for a velocity $\upsilon = \sqrt{2}\beta$, which corresponds to when there is no exponential growth in the displacement jumps, the dimensionless length $a_CM/(2\mu)$ of the zone of weakening is 0.422, which is only slightly less than the above mentioned static value $a_C(0)M/(2\mu) = 0.485$. It is clear that, in the case when the zone of weakening has a finite size, a transition from a rupture velocity which is less than the velocity of the transverse waves ($\upsilon < \beta$) to a velocity which exceeds that of the transverse waves ($\beta < \upsilon < \alpha$) can only be accomplished by abruptly.

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REFERENCES

1. SLEPYAN, L. I., *The Crack Mechanics*. Sudostroyeniye, Leningrad, 1981.
2. CHEN, K. C., HUANG, B.-S. and WANG, J.-H. *et al.*, An observation of rupture pulses of the 20 September 1999 Chi-Chi, Taiwan, earthquake from near-field seismograms. *Bull. Seism. Soc. America*, 2001, **91**, 1247–1254.

3. ANDREWS, D. J., Dynamic plane-strain shear rupture with a slip-weakening friction law calculated by a boundary integral method. *Bull. Seism. Soc. America*, 1985, **75**, 1–21.
4. DALGUER, L. A., IRIKURA, K. , RIERA, J. D. and CHIU, H. C., The importance of the dynamic source effects on strong ground motion during the 1999 Chi-Chi, Taiwan, earthquake: brief interpretation of damage distribution on building. *Bull. Seism. Soc. America*, 2001, **91**, 1112–1127.
5. NOVOZHILOV, V. V., The necessary and sufficient criterion of cohesive strength. *Prikl. Mat. Mekh.*, 1969, **33**, 2, 212–222.
6. MOROZOV, N. F., PETROV, Yu. V. and UTKIN, A. A., Calculation of the limit intensity of pulsed dynamic loads in crack mechanics. *Izv. Akad. Nauk SSSR, MTT*, 1988, 5, 180–182.
7. MOROZOV, N. F. and PETROV, Yu. V., A structure-time description of the velocity dependence of the dynamic viscosity of the fracture of brittle materials. *Izv. Ross. Akad. Nauk, MTT*, 1993, 6, 100–104.
8. PALMER, A. C. and RICE, J. R., The growth of slip in the progressive failure of over-consolidated clay. *Proc. Roy. Soc. London, Ser. A.*, 1973, **332**, 1591, 527–548.
9. GALIN, A. A., *Contact Problems in the Theory of Elasticity*, Gostekhizdst, Moscow, 1974.
10. CHEREPANOV, G. P., *Mechanics of Brittle Fracture*. Nauka, Moscow, 1974.
11. SHAPERY, R. A., A theory of crack initiation and growth in viscoelastic media. I. *Intern. J. Fracture*, 1975, **11**, 141–159.
12. LIN'KOV, A. M. and TLEUZHANOV, M. A., Calculation of local zones of irreversible deformation at the crack tip. *Izv. Akad. Nauk KirgSSR. Fiz.-Tekhn. i Mat. Nauki*, 1990, 1, 47–51.
13. LINKOV, A. M., Boundary value problem for crack growth in viscoelastic media. *Intern. J. Fracture*, 1974, **65**, 197–218.
14. ANDERSSON, H. and BERGKVIST, H., Analysis of a non-linear crack model. *J. Mech. Physics Solids*, 1970, **18**, 1–28.
15. IDA, Y., Cohesive force across the tip of a longitudinal shear crack and Griffith's specific surface energy. *J. Geophys. Research*, 1972, **77**, 20, 3796–3805.
16. ANDREWS, D. J., Rupture velocity of plane strain shear cracks. *J. Geophys. Research*, 1976, **81**, 32, 5679–5687.
17. LINKOV, A. M., Instability, fracture acceleration and wave amplification. *Intern. J. Rock Mech. Mining Sci.*, 2000, **37**, 31–37.
18. BELOV, Ye. B. and LIN'KOV, A. M., Stability conditions accompanying weakening in the interacting surfaces of cracks. In *Investigations on the Mechanics of Structural Designs and Materials* (Edited by V. D. Kharlab), St Petersburg. Gos. Arkhitekturno-Stroit. Univ., St Petersburg, 1995, 86–92.

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